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An Optimal Test for Variance Components of Multivariate Mixed-Effects Linear Models

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Abstract

In this article we derive an optimal test for testing the significance of covariance matrices of random-effects of two multivariate mixed-effects linear models. We compute the power of this newly derived test via simulation for various alternative hypotheses in a bivariate set up for unbalanced designs and observe that power responds sharply when sample size and alternative hypotheses are changed. For some balanced designs we compare power of the optimal test to that of the likelihood ratio test via simulation, and find that the proposed test has greater power than the likelihood ratio test. The results are illustrated using real data on human growth. Other relevant applications of the model are highlighted.

Keywords

Likelihood ratio test (LRT); locally best invariant test (LBI); growth curve models; unbalanced designs

1. Introduction

Mixed-effects regression models enjoy widespread utilization in fields ranging from mental health research to environmental statistics. The ability to accommodate the nesting of observations within experimental units has permitted a wide variety of applications of mixed-effects regression models. In the present context, we consider the case in which repeated observations are nested within individuals, which is typical of longitudinal studies of growth or the efficacy of medical interventions. In many applications, the outcome variable has multiple components, and the joint modeling of these components using multivariate extensions of the mixed-effects regression model is necessary for providing statistically rigorous tests of hypotheses. Such a model is known as a multivariate linear mixed-effects regression model. Conceptually, the random-effects represent the collective effect of unmeasured variables that contribute to the difference in the observed responses from subject to subject, above and beyond those effects associated with the fixed-effects in the model. The motivation for using a multivariate mixed-effects model is that it incorporates the correlation among the *p* outcome measures, which is ignored when performing a series of piecemeal univariate analyses. The net result is more realistic tests of hypotheses and interval estimates as compared to a series of simple univariate analyses. As an example, Dahm, Melton and Fuller (1983) have used multivariate mixed-effects models in analysis of animal breeding experiments, to draw inferences concerning an underlying genotype covariance matrix. Multivariate one-way random-effects models with random treatment effects, multivariate block designs with fixed block-effects, and random treatment effects are some examples of multivariate mixed-effects models. The fundamental problems are (i) fitting the model appropriately, (ii) estimating model parameters; both fixed and covariance matrices of the random-effects, and (iii) testing hypotheses regarding both fixed-effects and random-effects in the model. The primary focus of this paper is on hypothesis testing for the random-effects.

Despite the widespread use of mixed-effects regression models, available methods for hypothesis testing are quite limited for both univariate and multivariate mixed-effects models. Often, the tests are based on large sample theory (e.g., Wald test or likelihood ratio, chi-square statistics). In rare cases, approximate small sample tests are used to test significance of fixed-effects in the model (Milliken and Johnson, 1984). Alternatively, for testing significance of random-effects, boundary value problems typically preclude use of large sample tests based on chi-square statistics since they violate the regularity assumptions that lead to a chi-square distribution for the likelihood ratio test statistic.

As an illustration, consider a prospective longitudinal randomized clinical trial in which subjects are randomly assigned to treatment and control conditions and repeatedly measured over the course of the study on a series of end-points that are hypothesized to be affected by the treatment of interest. An example might be the comparison of novel and traditional antidepressant medications in the treatment of subjects with depression, where the end-points are four primary indications of the depressed state (e.g., depressed mood, anxiety, somatization, and sleep disorder).

In practice, we would typically analyze these data with a multivariate mixed-effects regression model, where the random-effects would include the intercept and slope of the regression of the response variables on time and the fixed-effects would include treatment and the treatment by time interaction. Testing would typically be restricted to determining the significance of the fixed-effects in the model. By contrast, in this article we develop a test to determine if the random-effects are non zero, and therefore needed to model the correlation of repeated measurements over time.

An Example: Growth Curve Models

As a typical example, consider two growth curve models, one for boys and the other for girls. We consider a study that took place in Sweden. Data from two registries in Sweden, the Medical Birth and the Patient's Registers, have been combined to identify children born as singletons in the city of Uppsala during 1973–1977 and to follow them regarding height(cm) and weight(kg) from medical records before and during school years through the age of 18 years. For a detailed description of these data, see Meng (1998), and Persson et. al., (1999). After log transformation, the data approximately follow a normal distribution. It can be shown that the log transformed height and weight data are approximately linear between one and ten years of age. The data are unbalanced as different children have different numbers of observations measured at different ages. Sun et. al., (2003) proposed bivariate unbalanced mixed-effects linear models for the log transformed height and weight measurements with fixed time-effects and random-effects for both subjects and errors. Sun et. al., (2003) derived some nonnegative estimators of variance components by generalizing the results of Mathew, Niyogi and Sinha (1994) under the assumptions that the random components are independently and normally distributed. We use these data to illustrate our methodology.

Previous Work

In practice, we use the likelihood ratio test or Wald's test to determine if under the null hypothesis a variance component is zero. This puts the variance component on the boundary of the parameter space defined by the alternative hypothesis. Under this scenario, the limiting distribution of the test statistic $-2(lnLR)$ under the null hypothesis does not follow a χ^2 distribution. Shapiro (1985) derived the asymptotic distribution of $-2(lnLR)$ as a mixture of χ^2 distributions, when the parameter under the null hypothesis falls on the boundary of a subset of the parametric space, but an interior point of the entire parametric space. Self and Liang (1987) generalized these results when the parameter under the null hypothesis is an interior point of the entire parametric space. Morrell (1998) applied a mixture of χ^2 distributions for likelihood ratio testing of variance components in the linear mixed-effects model using restricted maximum likelihood estimates. The test that we develop in this article does not suffer from the boundary value problem, and can be used as a small sample alternative to the LR chi-square statistic for testing variance components for normal theory mixed-effects regression models.

Khuri, Mathew and Sinha (1998) provide techniques based on small samples to derive optimal tests for variance components (i.e, covariance matrices for multivariate data) using Wijsman's (1967) representation theorem by exploiting the normality assumptions on both random-effects and error components. In the presence of nuisance parameters, when it is impossible to derive an optimal test for a fixed level of significance, Khuri, Mathew, and Sinha (1998) discuss some approximate tests like the generalized p-value approach of Tsui and Weerahandi (1989), Satterthwaite's (1941, 1946) approximation, and tests of Bartlett (1936) and Scheffe (1956) etc. In the context of multivariate mixed-effects models, Das and Sinha (1988), Mathew (1989), Mathew and Sinha (1988a, 1988b, 1992), Zhou and Mathew (1993, 1994) address some testing problems concerning the variance components corresponding to random-effects. The purpose of this article is to develop a testing procedure for matrix variance components of multivariate mixed-effects models.

Organization of the Paper

The paper is organized as follows. In Section 2, we propose a very general multivariate mixed-effects linear model for our study with only one vector of random components in addition to the random error components. We first describe covariance matrices, design matrices and various model parameters, and then develop the statistical foundation for our investigation. In Section 3, we define the steps needed to compare the two groups following the guidelines that (i) the outcome vector nested within a group can be modeled with an appropriate multivariate mixed-effects linear model, and (ii) subjects nested within and between groups are independent, and (iii) all the random components are independently and normally distributed. Exploiting the assumptions of normality, we derive a locally best invariant (LBI) test for the null hypothesis that the covariance matrices of the random components of both groups are equal to the null matrix under the assumption that the error covariance matrices of both groups are equal. To examine the performance of the LBI test, we simulate its power for various alternative hypotheses in Section 4. In section 5, we compare the power of the LBI test to that of the likelihood ratio test (LRT) for various alternative hypotheses via simulation. In Section 6, we illustrate our results with the growth curve model example. In Section 7, we provide a discussion.

2. Statistical Foundation

Let us assume that our experiment consists of two independent groups, and we have N_{ij} observations from the *jth* subject nested within the *ith* group. We further assume that there are n_i subjects in the *ith* group, and the total number of observations from the n_i subjects is N_i, where $N_i=\sum_{j=1}^{n_i} N_{ij}$, and $i = 1, 2$. Let

$$\boldsymbol{Y}=\begin{pmatrix}\boldsymbol{Y}_1\\ \boldsymbol{Y}_2\end{pmatrix},\ \mathbf{A}=\begin{pmatrix}\mathbf{A}_1 & \mathbf{0}\\ \mathbf{0} & \mathbf{A}_2\end{pmatrix},\ \boldsymbol{\Delta}=\begin{pmatrix}\boldsymbol{\Delta}_1\\ \boldsymbol{\Delta}_2\end{pmatrix},$$
$$\mathbf{X}=\begin{pmatrix}\mathbf{X}_1 & \mathbf{0}\\ \mathbf{0} & \mathbf{X}_2\end{pmatrix},\ \boldsymbol{\Theta}=\begin{pmatrix}\boldsymbol{\Theta}_1\\ \boldsymbol{\Theta}_2\end{pmatrix},\ \mathbf{E}=\begin{pmatrix}\mathbf{E}_1\\ \mathbf{E}_2\end{pmatrix}, \quad (1)$$

where for the *ith* group, $\mathbf{Y_i}$ is an $N_i \times p$ observation matrix, $\mathbf{A_i}$ is a known $N_i \times r_i$ covariate matrix, $\Delta_\mathbf{i}$ is a $r_i \times p$ fixed parameter matrix, $\mathbf{X_i}$ is a known $N_i \times s_i$ design matrix for the random effects, $\Theta_\mathbf{i}$ is a $s_i \times p$ random parameter matrix whose rows are independently distributed as $N(\mathbf{0}, \Sigma_{\Theta_\mathbf{i}})$, and $\mathbf{E_i}$ is the error matrix whose rows are independently distributed as $N(\mathbf{0}, \Sigma)$. $\Sigma_{\Theta_\mathbf{i}}$ is an unknown positive semidefinite matrix, and Σ is an unknown positive definite matrix. We assume that the distributions of Θ and $\mathbf{E}$ are independent. Thus the mixed-effects model for the two groups has the following expression

$$\boldsymbol{Y}=\mathbf{A}\boldsymbol{\Delta}+\mathbf{X}\boldsymbol{\Theta}+\mathbf{E}. \quad (2)$$

Let

$$\mathbf{Vec}(\boldsymbol{\Theta})=\begin{pmatrix}\mathbf{vec}(\boldsymbol{\Theta}_1)\\ \mathbf{vec}(\boldsymbol{\Theta}_2)\end{pmatrix}. \quad \text{Hence } Cov(Vec(\boldsymbol{\Theta}'))=\begin{pmatrix}\mathbf{I}_{\mathbf{s_1}}\otimes\sum_{\boldsymbol{\Theta}_1} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}_{\mathbf{s_2}}\otimes\sum_{\boldsymbol{\Theta}_2}\end{pmatrix}. \quad (3)$$

3. Significance of Random-Effects

In practice, an experimenter will prefer to use a mixed-effects model to a fixed-effects model only if the contribution due to the random components is significant. In the two group case, this involves simultaneous testing that both random-effects covariance matrices are zero, i.e.,

$$H_0:\sum_{\boldsymbol{\Theta}_1}=\sum_{\boldsymbol{\Theta}_2}=\mathbf{0}. \quad (4)$$

The implication of the hypothesis H_0 in (4) is that the random components of the model (except the error term) are not significant, hence upon acceptance of H_0, a fixed-effects model is sufficient to explain the linear relationship between the outcome variables and covariates of interest.

In the context of a multivariate mixed-effects regression model, testing that two random-effects covariance matrices are equal to the null matrix (i.e., H_0) for finite samples is an open problem. We seek a locally best invariant test for H_0 using Wijsman's representation theorem and some other techniques suggested by Zhou and Mathew (1993), and Khuri, Mathew and Sinha (1998).

Let $N = N_1 + N_2$, $r = r_1 + r_2$, and $s = s_1 + s_2$. Note that $Vec(\mathbf{Y}') = Vec(\Delta'\mathbf{A}') + \mathbf{Vec}(\Theta'\mathbf{X}') + \mathbf{Vec}(\mathbf{E}')$. Hence

$$Cov(Vec(\mathbf{Y}')) = \begin{pmatrix} \mathbf{V_1} \otimes \mathbf{E_{\Theta_1}} & \mathbf{0} \\ \mathbf{0} & \mathbf{V_2} \otimes \sum_{\Theta_2} \end{pmatrix} + \mathbf{I_N} \otimes \sum, \quad (5)$$

where $\mathbf{V_i} = \mathbf{X_i X_i'}$, for $i = 1, 2$. Let $N_i > r_i$, and $\mathbf{A_i}$ be a full row rank matrix. Let $\mathbf{Z_i}$ be an $N_i \times (N_i - r_i)$ matrix such that $\mathbf{Z_i'A_i} = \mathbf{0}$, and $\mathbf{Z_i'Z_i} = \mathbf{I}_{N_i - r_i}$. This implies that the column space of $\mathbf{Z_i}$ is orthogonal to the column space of $\mathbf{A_i}$, and the error space is defined with the help of the matrix $\mathbf{Z_i}$. Let

$$\mathbf{Z} = \begin{pmatrix} \mathbf{Z_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{Z_2} \end{pmatrix}, \text{ and } \mathbf{U} = \begin{pmatrix} \mathbf{U_1} \\ \mathbf{U_2} \end{pmatrix} = \mathbf{Z}'\mathbf{Y} = \begin{pmatrix} \mathbf{Z_1'Y_1} \\ \mathbf{Z_2'Y_2} \end{pmatrix}, \quad (6)$$

where $\mathbf{Z}$ is an $N \times (N - r)$ matrix, $\mathbf{U_1}$ is an $(N_1 - r_1) \times p$, $\mathbf{U_2}$ is an $(N_2 - r_2) \times p$ matrices. Note that $E(\mathbf{U}) = 0$, and

$$Cov(Vec(\mathbf{U}')) = \begin{pmatrix} \mathbf{V_{11}} \otimes \sum_{\Theta_1} + \mathbf{I_{N_1 - r_1}} \otimes \sum & \mathbf{0} \\ \mathbf{0} & \mathbf{V_{22}} \otimes \sum_{\Theta_2} + \mathbf{I_{N_2 - r_2}} \otimes \sum \end{pmatrix}, \quad (7)$$

where $\mathbf{V_{ii}} = \mathbf{Z_i'V_iZ_i}$. Let $\lambda_{i1} >, \cdots, > \lambda_{ig_i}$ be the ordered distinct nonzero eigen values of $\mathbf{V_{ii}}$. Let $\mathbf{V_{ii}} = \sum_{j=1}^{g_i} \lambda_{ij} \mathbf{F_{ij}}$ be the spectral decomposition of $\mathbf{V_{ii}}$. Let $\mathbf{F_{i(g+1)}} = I - \sum_{j=1}^{g_i} \mathbf{F_{ij}}$. The density function of $Vec(\mathbf{U}')$ denoted by $f(\mathbf{U})$ has the following expression

$$\begin{aligned} f(\mathbf{U}) = & |\mathbf{V_{11}} \otimes \sum_{\Theta_1} + \mathbf{I_{N_1 - r_1}} \otimes \sum|^{-1/2} |\mathbf{V_{22}} \otimes \sum_{\Theta_2} + \mathbf{I_{N_2 - r_2}} \otimes \sum|^{-1/2} \\ & \times exp\{-\frac{1}{2}\left[\sum_{j=1}^{g_1} tr\,(\lambda_{1j} \sum_{\Theta_1} + \sum)^{-1} \mathbf{U_1'F_{1j}U_1} + tr(\sum^{-1} \mathbf{U_1'F_{1(g+1)}U_1})\right]\} \\ & \times exp\{-\frac{1}{2}\left[\sum_{j=1}^{g_2} tr(\lambda_{2j} \sum_{\Theta_2} + \sum)^{-1} \mathbf{U_2'F_{2j}U_2} + tr(\sum^{-1} \mathbf{U_2'F_{2(g+1)}U_2})\right]\}. \end{aligned} \quad (8)$$

Note that the testing problem $H_0 : \Sigma_{\Theta_1} = \Sigma_{\Theta_2} = 0$, against the alternative that $H_a : \Sigma_{\Theta_1} \neq 0$, $\Sigma_{\Theta_2} \neq 0$ remains left invariant under the group $GL(p)$ of $p \times p$ nonsingular matrices acting on $\mathbf{U}$ as $\mathbf{U} \to \mathbf{UW}'$, where $\mathbf{W}$ is a nonsingular matrix. The Jacobian of this transformation is $\mathbf{J} = |\mathbf{W'W}|^{(\mathbf{N}-\mathbf{r})/2}$. A left invariant measure on $GL(p)$ is $|\mathbf{W'W}|^{\mathbf{p}/2}$. Applying Wijsman's (1968) representation theorem, the ratio R of the nonnull to the null distribution of a maximal invariant is given by

$$R = \frac{R_1}{R_0} = \frac{\int_{\mathbf{GL(p)}} f(\mathbf{UW}/H_{a1}) |\mathbf{W'W}|^{(N-r-p)/2} d\mathbf{W}}{\int_{\mathbf{GL(p)}} f(\mathbf{UW}/H_{01}) |\mathbf{W'W}|^{(N-r-p)/2} d\mathbf{W}}, \quad (9)$$

where $f(\mathbf{U}/H_l)$ denotes the density of $\mathbf{U}$ under H_l, for $l = 0, a$. Let $\sum_{\Theta_i}^{*} = \sum^{-1/2} \sum_{\Theta_i} \sum^{-1/2}$. For evaluating the integral, assume without any loss of generality that $\Sigma = \mathbf{I}$. Using the expression of the density function of $\mathbf{U}$ given in (8), we now compute the numerator of R as

$$R_1=|\mathbf{V_{11}}\otimes\sum\nolimits_{\mathbf{\Theta_1}}^{*}+\mathbf{I_{N_1-r_1}}\otimes\mathbf{I_p}|^{-\frac{1}{2}}|\mathbf{V_{22}}\otimes\sum\nolimits_{\mathbf{\Theta_2}}^{*}+\mathbf{I_{N_2-r_2}}\otimes\mathbf{I_p}|^{\frac{-1}{2}}$$
$$\times\int_{\mathbf{GL}(p)}exp\{-\tfrac{1}{2}\left[\sum_{j=1}^{g_1}tr(\lambda_{1j}\sum\nolimits_{\mathbf{\Theta_1}}^{*}+\mathbf{I_p})^{-1}\mathbf{W}'\mathbf{U}_1'\mathbf{F_{1j}}\mathbf{U_1}\mathbf{W}+tr(\mathbf{W}'\mathbf{U}_1'\mathbf{F_{1(g+1)}}\mathbf{U_1}\mathbf{W})\right]\}$$
$$\times exp\{-\tfrac{1}{2}\left[\sum_{j=1}^{g_2}tr(\lambda_{2j}\sum\nolimits_{\mathbf{\Theta_2}}^{*}+\mathbf{I_p})^{-1}\mathbf{W}'\mathbf{U}_2'\mathbf{F_{2j}}\mathbf{U_2}\mathbf{W}+tr(\mathbf{W}'\mathbf{U}_2'\mathbf{F_{2(g+1)}}\mathbf{U_2}\mathbf{W})\right]\}$$
$$|\mathbf{W}'\mathbf{W}|^{\frac{N-r-p}{2}}d\mathbf{W}. \tag{10}$$

From (10) we see that no uniformly most powerful invariant unbiased test exists. We derive a locally best invariant test by expanding both $\sum_{\mathbf{\Theta_1}}^{*}$ and $\sum_{\mathbf{\Theta_2}}^{*}$ in the neighborhood of zero. Let $\sum_{\mathbf{ij}}=(\lambda_{ij}\sum_{\mathbf{\Theta_i}}^{*}+\mathbf{I_p})^{-1}$. Hence expanding the exponentials in (10) by Taylor's series, the expression of R_1 in terms of Σ_{ij}*'s* is

$$R_1=|\mathbf{V_{11}}\otimes\sum\nolimits_{\mathbf{\Theta_1}}^{*}+\mathbf{I_{N_1-r_1}}\otimes\mathbf{I_p}|^{\frac{-1}{2}}|\mathbf{V_{22}}\otimes\sum\nolimits_{\mathbf{\Theta_2}}^{*}+\mathbf{I_{N_2-r_2}}\otimes\mathbf{I_p}|^{-\frac{1}{2}}$$
$$\times\int_{\mathbf{GL}(p)}exp\{tr(-\tfrac{1}{2}\mathbf{W}'\mathbf{U}_1'\mathbf{F_{1(g+1)}}\mathbf{U_1}\mathbf{W}+\mathbf{W}'\mathbf{U}_2'\mathbf{F_{2(g+1)}}\mathbf{U_2}\mathbf{W})\}$$
$$\times\left(\prod_{j=1}^{g_1}exp\{tr(-\tfrac{1}{2}[\mathbf{W}'\mathbf{U}_1'\mathbf{F_{1j}}\mathbf{U_1}\mathbf{W}])\}\right)\left(\prod_{j=1}^{g_2}exp\{tr(-\tfrac{1}{2}[\mathbf{W}'\mathbf{U}_2'\mathbf{F_{2j}}\mathbf{U_2}\mathbf{W}])\}\right)$$
$$\times\left(\sum_{l=0}^{\infty}\tfrac{1}{l!}\left[-\tfrac{1}{2}tr([\sum\nolimits_{\mathbf{1j}}-\mathbf{I}_p]\mathbf{W}'\mathbf{U}_1'\mathbf{F_{1j}}\mathbf{U_1}\mathbf{W})\right]^l\right)$$
$$\times\left(\sum_{l=0}^{\infty}\tfrac{1}{l!}\left[-\tfrac{1}{2}tr([\sum\nolimits_{\mathbf{2j}}-\mathbf{I}_p]\mathbf{W}'\mathbf{U}_2'\mathbf{F_{2j}}\mathbf{U_2}\mathbf{W})\right]^l\right)|\mathbf{W}'\mathbf{W}|^{\frac{N-r-p}{2}}d\mathbf{W}. \tag{11}$$

Write $\mathbf{W_1}=(\mathbf{U}_1'\mathbf{U_1})^{\frac{1}{2}}\mathbf{W},\mathbf{W_2}=(\mathbf{U}_2'\mathbf{U_2})^{\frac{1}{2}}\mathbf{W},\mathbf{K_1}=\mathbf{U_1}(\mathbf{U}_1'\mathbf{U_1})^{-1/2},\mathbf{K_2}=\mathbf{U_2}(\mathbf{U}_2'\mathbf{U_2})^{-1/2}$. Note that $\mathbf{K}_1'\mathbf{K_1}=\mathbf{I_p},\mathbf{K}_2'\mathbf{K_2}=\mathbf{I_p},\sum_{j=1}^{g_i+1}\mathbf{F_{ij}}=\mathbf{I_p}$, i = 1, 2. Using these expressions, we simplify R_1 as

$$R_1=|\mathbf{V_{11}}\otimes\sum\nolimits_{\mathbf{\Theta_1}}^{*}+\mathbf{I_{N_1-r_1}}\otimes\mathbf{I_p}|^{\frac{-1}{2}}|\mathbf{V_{22}}\otimes\sum\nolimits_{\mathbf{\Theta_2}}^{*}+\mathbf{I_{N_2-r_2}}\otimes\mathbf{I_p}|^{\frac{-1}{2}}$$
$$\times\int_{\mathbf{GL}(p)}exp\{tr(-\tfrac{1}{2}[\mathbf{W}_1'\mathbf{W_1}+\mathbf{W}_2'\mathbf{W_2}])\}$$
$$\times\left(\prod_{j=1}^{g_1}\sum_{l=0}^{\infty}\tfrac{1}{l!}\left[-\tfrac{1}{2}tr([\sum\nolimits_{\mathbf{1j}}-\mathbf{I_p}]\mathbf{W}_1'\mathbf{K}_1'\mathbf{F_{1j}}\mathbf{K_1}\mathbf{W_1})\right]^l\right)$$
$$\times\left(\prod_{j=1}^{g_2}\sum_{l=0}^{\infty}\tfrac{1}{l!}\left[-\tfrac{1}{2}tr([\sum\nolimits_{\mathbf{2j}}-\mathbf{I_p}]\mathbf{W}_2'\mathbf{K}_2'\mathbf{F_{2j}}\mathbf{K_2}\mathbf{W_2})\right]^l\right)|\mathbf{W}'\mathbf{W}|^{\frac{N-r-p}{2}}d\mathbf{W} \tag{12}$$

In order to further simplify (12) we use some inequalities on the trace of a matrix. Let

$$a_l^{ij}=\tfrac{1}{l!}\left[-\tfrac{1}{2}tr([\sum\nolimits_{\mathbf{ij}}-\mathbf{I}_p]\mathbf{W}'\mathbf{U}_\mathbf{i}'\mathbf{F_{ij}}\mathbf{U_i}\mathbf{W})\right]^l$$
$$|a_l^{ij}|\le\tfrac{1}{2^l l!}[tr(\mathbf{I_p}-\sum\nolimits_{\mathbf{ij}})^2]^{l/2}[tr(\mathbf{W}'\mathbf{U}_\mathbf{i}'\mathbf{F_{ij}}\mathbf{U}_\mathbf{i}'\mathbf{W})^2]^{l/2}$$
$$\le\tfrac{1}{2^l l!}\lambda_{ij}^l[tr(\sum\nolimits_{\mathbf{\Theta_i}}^{*\mathbf{2}})]^{l/2}[tr(\mathbf{W}'\mathbf{U}_\mathbf{ij}'\mathbf{U_i}\mathbf{W})^2]^{l/2}. \tag{13}$$

The second inequality in (13) follows from inequalities $\mathbf{K}_\mathbf{i}'\mathbf{F_{ij}}\mathbf{K_i}\le\mathbf{I_p}$, and the fact that $\mathbf{I_p}-\sum_{\mathbf{ij}}\le\lambda_{ij}\sum_{\mathbf{\Theta_i}}^{*}$. Now use the fact that $tr\,([\sum_{\mathbf{\Theta_i}}^{*}]^2)\le[tr(\sum_{\mathbf{\Theta_i}}^{*})]^2$ to simplify

$$|a_l^{ij}| \leq \frac{1}{2^l l!}\lambda_{ij}^l [tr(\sum\nolimits_{\boldsymbol{\Theta}_\mathbf{i}}^*)]^l [tr(\mathbf{W}'\mathbf{U}_\mathbf{ij}'\mathbf{U}_\mathbf{i}\mathbf{W})^2]^{l/2}. \quad (14)$$

Hence from (14) we obtain

$$\begin{aligned} |a_l^{ij}| &= [tr(\mathbf{W}'\mathbf{U}_\mathbf{i}'\mathbf{U}_\mathbf{i}\mathbf{W})^2]^{l/2} o(tr(\sum\nolimits_{\boldsymbol{\Theta}_\mathbf{i}}^*)),\ l \geq 2 \\ |a_1^{ij} a_1^{ij'}| &= [tr(\mathbf{W}'\mathbf{U}_\mathbf{i}'\mathbf{U}_\mathbf{i}\mathbf{W})^2]^{l/2} o(tr(\sum\nolimits_{\boldsymbol{\Theta}_\mathbf{i}}^*)), \end{aligned} \quad (15)$$

uniformly in the data, as $\sum_{\boldsymbol{\Theta}_\mathbf{i}}^* \to 0$. Using these results we simplify the expression of R_1 in (12) as follows.

$$\begin{aligned} R_1 = &|\mathbf{V_{11}} \otimes \sum\nolimits_{\boldsymbol{\Theta}_\mathbf{1}}^* + \mathbf{I}_{\mathbf{N_1 - r_1}} \otimes \mathbf{I_p}|^{-\frac{1}{2}} |\mathbf{V_{22}} \otimes \sum\nolimits_{\boldsymbol{\Theta}_\mathbf{2}}^* + \mathbf{I}_{\mathbf{N_2 - r_2}} \otimes \mathbf{I_p}|^{-\frac{1}{2}} \\ &\times \int_{\mathbf{GL}(p)} exp\{tr(-\tfrac{1}{2}[\mathbf{W}'\mathbf{U}_\mathbf{1}'\mathbf{U_1}\mathbf{W} + \mathbf{W}'\mathbf{U}_\mathbf{2}'\mathbf{U_2}\mathbf{W}])\} \\ &\times \left(1 + \tfrac{1}{2}\sum_{j=1}^{g_1} tr([\mathbf{I_p} - \sum\nolimits_{\mathbf{1j}}]\mathbf{W}'\mathbf{U}_\mathbf{1}'\mathbf{F_{1j}}\mathbf{U_1}\mathbf{W})\right)\left(1 + \tfrac{1}{2}\sum_{j=1}^{g_2} tr([\mathbf{I_p} - \sum\nolimits_{\mathbf{2j}}]\mathbf{W}'\mathbf{U}_\mathbf{2}'\mathbf{F_{2j}}\mathbf{U_2}\mathbf{W})\right) \\ &\times |\mathbf{W}'\mathbf{W}|^{\frac{N-r-p}{2}} d\mathbf{W} + o(tr(\sum\nolimits_{\boldsymbol{\Theta}_\mathbf{1}}^*), tr(\sum\nolimits_{\boldsymbol{\Theta}_\mathbf{1}}^*)). \end{aligned} \quad (16)$$

In order to further simplify (16), we use a result (Lemma 2.5) of Kariya and Sinha (1989). Let **O** be an orthogonal matrix of dimension $p \times p$, then

$\int_{\mathbf{O}(p)} tr(\mathbf{A_1}\mathbf{O}\mathbf{A_2}\mathbf{O}')d\mathbf{O} = \frac{1}{p}[tr(\mathbf{A_1})][tr(\mathbf{A_2})]$, where $d\mathbf{O}$ denotes the uniform distribution on $\mathbf{O}(p)$. We use this result in (16) and obtain the following.

$$\begin{aligned} R_1 = &|\mathbf{V_{11}} \otimes \sum\nolimits_{\boldsymbol{\Theta}_\mathbf{1}}^* + \mathbf{I}_{\mathbf{N_1 - r_1}} \otimes \mathbf{I_p}|^{-\frac{1}{2}} |\mathbf{V_{22}} \otimes \sum\nolimits_{\boldsymbol{\Theta}_\mathbf{2}}^* + \mathbf{I}_{\mathbf{N_2 - r_2}} \otimes \mathbf{I_p}|^{\frac{-1}{2}} \\ &\times \int_{\mathbf{GL}(p)} exp\{tr(-\tfrac{1}{2}[\mathbf{W}'\mathbf{U}_\mathbf{1}'\mathbf{U_1}\mathbf{W} + \mathbf{W}'\mathbf{U}_\mathbf{2}'\mathbf{U_2}\mathbf{W}])\} \\ &\times \left[1 + \tfrac{1}{2p^2}\sum_{j=1}^{g_1} tr(\mathbf{I_p} - \sum\nolimits_{\mathbf{1j}}) tr(\mathbf{W}'\mathbf{W}) tr(\mathbf{U}_\mathbf{1}'\mathbf{F_{1j}}\mathbf{U_1})\right] \\ &\times \left[1 + \tfrac{1}{2p^2}\sum_{j=1}^{g_2} tr(\mathbf{I_p} - \sum\nolimits_{\mathbf{2j}}) tr(\mathbf{W}'\mathbf{W}) tr(\mathbf{U}_\mathbf{2}'\mathbf{F_{2j}}\mathbf{U_2})\right] \\ &\times |\mathbf{W}'\mathbf{W}|^{\frac{N-r-p}{2}} d\mathbf{W} + o\left(tr(\sum\nolimits_{\boldsymbol{\Theta}_\mathbf{1}}^*), tr(\sum\nolimits_{\boldsymbol{\Theta}_\mathbf{1}}^*)\right). \end{aligned} \quad (17)$$

Let $\mathbf{B}_{ij} = \lambda_{ij}\sum_{\boldsymbol{\Theta}_\mathbf{1}}^*$. Note that

$$\begin{aligned} tr(\mathbf{I}_p - \sum\nolimits_{ij}) &= tr\,(\mathbf{I}_p - (\mathbf{B}_{ij} + \mathbf{I}_p)^{-1}) \\ &= tr\,(\mathbf{I}_p - [\mathbf{I}_p - \mathbf{B}_{ij}(\mathbf{B}_{ij} + \mathbf{B}_{ij}\mathbf{B}_{ij})^{-1}\mathbf{B}_{ij}]) \\ &= tr\,\left(\lambda_{ij}\sum\nolimits_{\boldsymbol{\Theta}_\mathbf{i}}^*(\mathbf{I}_p + \lambda_{ij}\sum\nolimits_{\boldsymbol{\Theta}_\mathbf{i}}^*)^{-1}\right) \\ &= \lambda_{ij} tr(\sum\nolimits_{\boldsymbol{\Theta}_\mathbf{i}}^*) + o(tr(\sum\nolimits_{\boldsymbol{\Theta}_\mathbf{i}}^*)). \end{aligned} \quad (18)$$

Using (18) in (17) we obtain

$$\begin{aligned}R_1=&|\mathbf{V_{11}}\otimes\textstyle\sum^*_{\mathbf{\Theta_1}}+\mathbf{I_{N_1-r_1}}\otimes\mathbf{I_p}|^{\frac{-1}{2}}|\mathbf{V_{22}}\otimes\textstyle\sum^*_{\mathbf{\Theta_2}}+\mathbf{I_{N_2-r_2}}\otimes\mathbf{I_p}|^{\frac{-1}{2}}\\&\times\int_{\mathbf{GL}(p)}exp\left\{tr(-\tfrac{1}{2}[\mathbf{W}'\mathbf{U_1'}\mathbf{U_1}\mathbf{W}+\mathbf{W}'\mathbf{U_2'}\mathbf{U_2}\mathbf{W}])\right\}\\&\times\left[1+\frac{1}{2p^2}\sum_{j=1}^{g_1}tr(\lambda_{1j}\textstyle\sum^*_{\mathbf{\Theta_1}})tr(\mathbf{W}'\mathbf{W})tr(\mathbf{U_1'}\mathbf{F_{1j}}\mathbf{U_1})\right]\\&\times\left[1+\frac{1}{2p^2}\sum_{j=1}^{g_2}tr(\lambda_{2j}\textstyle\sum^*_{\mathbf{\Theta_2}})tr(\mathbf{W}'\mathbf{W})tr(\mathbf{U_2'}\mathbf{F_{2j}}\mathbf{U_2})\right]\\&\times|\mathbf{W}'\mathbf{W}|^{\frac{N-r-p}{2}}d\mathbf{W}+o(tr(\textstyle\sum^*_{\mathbf{\Theta_1}}),tr(\textstyle\sum^*_{\mathbf{\Theta_1}})).\end{aligned}\quad(19)$$

Using $\mathbf{V_{11}}=\sum_{j=1}^{g_1}\lambda_{1j}\mathbf{F_{ij}}$, and $\mathbf{V_{22}}=\sum_{j=1}^{g_2}\lambda_{2j}\mathbf{F_{2j}}$ in (19) we obtain

$$\begin{aligned}R_1=&|\mathbf{V_{11}}\otimes\textstyle\sum^*_{\mathbf{\Theta_1}}+\mathbf{I_{N_1-r_1}}\otimes\mathbf{I_p}|^{\frac{-1}{2}}|\mathbf{V_{22}}\otimes\textstyle\sum^*_{\mathbf{\Theta_2}}+\mathbf{I_{N_2-r_2}}\otimes\mathbf{I_p}|^{\frac{-1}{2}}\\&\times\int_{\mathbf{GL}(p)}exp\left\{tr(-\tfrac{1}{2}[\mathbf{W}'\mathbf{U_1'}\mathbf{U_1}\mathbf{W}+\mathbf{W}'\mathbf{U_2'}\mathbf{U_2}\mathbf{W}])\right\}\\&\times\left[1+\tfrac{1}{2p^2}tr(\textstyle\sum^*_{\mathbf{\Theta_1}})tr(\mathbf{W}'\mathbf{W})tr(\mathbf{U_1'}\mathbf{V_{11}}\mathbf{U_1})\right]\\&\times\left[1+\tfrac{1}{2p^2}tr(\textstyle\sum^*_{\mathbf{\Theta_2}})tr(\mathbf{W}'\mathbf{W})tr(\mathbf{U_2'}\mathbf{V_{22}}\mathbf{U_2})\right]\\&\times|\mathbf{W}'\mathbf{W}|^{\frac{N-r-p}{2}}d\mathbf{W}+o(tr(\textstyle\sum^*_{\mathbf{\Theta_1}}),tr(\textstyle\sum^*_{\mathbf{\Theta_1}})).\end{aligned}\quad(20)$$

Next, we use some determinant approximations in the neighborhood of the zero of $\sum^*_{\mathbf{\Theta_i}}$.

$$\begin{aligned}&|\mathbf{V_{ii}}\otimes\textstyle\sum^*_{\mathbf{\Theta_i}}+\mathbf{I_{N_i-r_i}}\otimes\mathbf{I_p}|^{\frac{-1}{2}}\simeq1-\tfrac{1}{2}(tr\textstyle\sum^*_{\mathbf{\Theta_i}})\,(tr\mathbf{V}_{ii}),\ \text{for i}=1,2.\ \text{Hence}\\&|\mathbf{V_{11}}\otimes\textstyle\sum^*_{\mathbf{\Theta_1}}+\mathbf{I_{N_1-r_1}}\otimes\mathbf{I_p}|^{\frac{-1}{2}}|\mathbf{V_{22}}\otimes\textstyle\sum^*_{\mathbf{\Theta_2}}+\mathbf{I_{N_2-r_2}}\otimes\mathbf{I_p}|^{\frac{-1}{2}}\\&\simeq1-\tfrac{1}{2}tr(\textstyle\sum^*_{\mathbf{\Theta_1}})tr(\mathbf{V}_{11})-\tfrac{1}{2}tr(\textstyle\sum^*_{\mathbf{\Theta_2}})tr(\mathbf{V}_{22}).\end{aligned}\quad(21)$$

Let

$$\mathbf{J}=\mathbf{U_1'}\mathbf{U_1}+\mathbf{U_2'}\mathbf{U_2},\quad(22)$$

$$\text{hence }\mathbf{W}'(\mathbf{U_1'}\mathbf{U_1}+\mathbf{U_2'}\mathbf{U_2})\mathbf{W}=\mathbf{W}'\mathbf{J}\mathbf{W}.\quad(23)$$

$$\mathbf{C}=1-\frac{1}{2}tr(\sum\nolimits^*_{\mathbf{\Theta_1}})tr(\mathbf{V}_{11})-\frac{1}{2}tr(\sum\nolimits^*_{\mathbf{\Theta_2}})tr(\mathbf{V}_{22}),\quad(24)$$

$$\mathbf{D}=\frac{1}{2p^2}tr(\sum\nolimits^*_{\mathbf{\Theta_1}})tr(\mathbf{U_1'}\mathbf{V_{11}}\mathbf{U_1})+\frac{1}{2p^2}tr(\sum\nolimits^*_{\mathbf{\Theta_2}})tr(\mathbf{U_2'}\mathbf{V_{22}}\mathbf{U_2}).\quad(25)$$

Note that when $\Sigma_{\mathbf{\Theta_1}}$ and $\Sigma_{\mathbf{\Theta_2}}$ are in the neighborhood of zero, **CD** ≃ **D**. Using this result and the symbols **J**, **C**, and **D** we express R_1 in (20) as

$$\begin{aligned}R_1\simeq&\ \mathbf{C}\int_{\mathbf{GL}(p)}exp\left\{tr(-\tfrac{1}{2}\mathbf{W}'\mathbf{J}\mathbf{W})\right\}|\mathbf{W}'\mathbf{W}|^{(N-p-r)/2}d\mathbf{W}\\&+\mathbf{D}\int_{\mathbf{GL}(p)}exp\{tr(-\tfrac{1}{2}\mathbf{W}'\mathbf{J}\mathbf{W})\}tr(\mathbf{W}'\mathbf{W})|\mathbf{W}'\mathbf{W}|^{(N-p-r)/2}d\mathbf{W}.\end{aligned}\quad(26)$$

Note that under H_{01}, $\mathbf{C}_0 = \mathbf{C}/(\Sigma_{\mathbf{\Theta_1}} = \Sigma_{\mathbf{\Theta_2}} = 0) = 1$, and $\mathbf{D_0} = \mathbf{D}/(\Sigma_{\mathbf{\Theta_1}} = \Sigma_{\mathbf{\Theta_2}} = 0) = 0$. Using these results, we get R_0, the similar expression as of R_1 under the null hypothesis.

$$R_0 \simeq \int_{\mathbf{GL}(p)} \exp\{tr(-\frac{1}{2}\mathbf{W}'\mathbf{J}\mathbf{W})\}|\mathbf{W}'\mathbf{W}|^{(N-p-r)/2} d\mathbf{W}. \quad (27)$$

Hence the ratio of the non-null to null is

$$R = \frac{R_1}{R_0} \simeq \mathbf{C} + \mathbf{D}\frac{\int_{\mathbf{GL}(p)} exp\{tr(-\frac{1}{2}\mathbf{W}'\mathbf{J}\mathbf{W})\} tr(\mathbf{W}'\mathbf{W})|\mathbf{W}'\mathbf{W}|^{(N-p-r)/2} d\mathbf{W}}{\int_{\mathbf{GL}(p)} exp\{tr(-\frac{1}{2}\mathbf{W}'\mathbf{J}\mathbf{W})\}|\mathbf{W}'\mathbf{W}|^{(N-p-r)/2} d\mathbf{W}}. \quad (28)$$

Let us now simplify the expression of the fraction lying in the right side of (28). Let $\mathbf{W_3} = \mathbf{J}^{1/2}\mathbf{W}$, then $d\mathbf{W_3} = |\mathbf{J}|^{p/2}\, d\mathbf{W}$, $\mathbf{WJW} = \mathbf{W_3}'\mathbf{W_3}$, and

$$tr(\mathbf{W}'\mathbf{W}) = \frac{tr(\mathbf{J}^{-1}) tr(\mathbf{W_3}'\mathbf{W_3})}{p}. \quad (29)$$

$$|\mathbf{W}'\mathbf{W}|^{(N-p-r)/2} = |\mathbf{J}^{-1}|^{(N-p-r)/2}|\mathbf{W_3}'\mathbf{W_3}|^{(N-p-r)/2}. \quad (30)$$

Using this new transformation and the expressions given in (29) and (30), we simplify below the non-null to null ratio in (28).

$$\begin{aligned} R &\simeq \mathbf{C} + \mathbf{D}\frac{\int_{\mathbf{GL}(p)} exp\{tr(-\frac{1}{2}\mathbf{W}'\mathbf{J}\mathbf{W})\} tr(\mathbf{W}'\mathbf{W})|\mathbf{W}'\mathbf{W}|^{(N-P-r)/2} d\mathbf{W}}{\int_{\mathbf{GL}(p)} \exp\{tr(-\frac{1}{2}\mathbf{W}'\mathbf{J}\mathbf{W})\}|\mathbf{W}'\mathbf{W}|^{(N-p-r)/2} d\mathbf{W}}, \\ &\simeq \mathbf{C} + \mathbf{D}\frac{\int_{\mathbf{GL}(p)} exp\{tr(-\frac{1}{2}\mathbf{W_3}'\mathbf{W_3})\} tr(\mathbf{J^{-1}})|\mathbf{J}|^{(-N+r)/2} tr(\mathbf{W_3}'\mathbf{W_3})|\mathbf{W_3}'\mathbf{W_3}|^{(N-p-r)/2} d\mathbf{W_3}}{\int_{\mathbf{GL}(p)} exp\{tr(-\frac{1}{2}\mathbf{W_3}'\mathbf{W_3})\}|\mathbf{J}|^{(-N+r)/2}|\mathbf{W_3}'\mathbf{W_3}|^{(N-p-r)/2} d\mathbf{W_3}}, \\ &\simeq \mathbf{C} + \mathbf{D} tr(\mathbf{J}^{-1}) c, \end{aligned} \quad (31)$$

where

$$\begin{aligned} c &= \frac{\int_{\mathbf{GL}(p)} exp\{tr(-\frac{1}{2}\mathbf{W_3}'\mathbf{W_3})\} tr(\mathbf{W_3}'\mathbf{W_3})|\mathbf{W_3}'\mathbf{W_3}|^{(N-p-r)/2} d\mathbf{W_3}}{\int_{\mathbf{GL}(p)} exp\{tr(-\frac{1}{2}\mathbf{W_3}'\mathbf{W_3})\}|\mathbf{W_3}'\mathbf{W_3}|^{(N-p-r)/2} d\mathbf{W_3}}, \\ &= \text{is a constant, does not depend on the data.} \end{aligned} \quad (32)$$

Thus the ratio R is simplified as

$$\begin{aligned} R &\simeq \mathbf{C} + \mathbf{D} tr(\mathbf{J}^{-1}) c, \\ &\simeq 1 - \tfrac{1}{2} tr(\textstyle\sum_{\mathbf{\Theta_1}}^*) tr(\mathbf{V_{11}}) - \tfrac{1}{2} tr(\textstyle\sum_{\mathbf{\Theta_2}}^*) tr(\mathbf{V_{22}}) + \tfrac{c}{2p^2}[tr(\textstyle\sum_{\mathbf{\Theta_1}}^*) tr(\mathbf{U_1}'\mathbf{V_{11}}\mathbf{U_1}) tr(\mathbf{J}^{-1}) + tr(\textstyle\sum_{\mathbf{\Theta_2}}^*) tr(\mathbf{U_2}'\mathbf{V_{22}}\mathbf{U_2}) tr(\mathbf{J}^{-1})]. \end{aligned} \quad (33)$$

Note that in (33) only $tr(\mathbf{U_1}'\mathbf{V_{11}}\mathbf{U_1})$, $tr(\mathbf{U_2}'\mathbf{V_{22}}\mathbf{U_2})$, and $tr(\mathbf{J}^{-1})$ are functions of the data. Under the alternative hypothesis when $\Sigma_{\mathbf{\Theta_1}}$ and $\Sigma_{\mathbf{\Theta_2}}$ are in the neighborhood of zero, a reasonable test is to reject H_0 when

$\mathbf{T_1}(\boldsymbol{Y}) = tr(\mathbf{U_1}'\mathbf{V_{11}}\mathbf{U_1}) tr(\mathbf{J}^{-1}) + tr(\mathbf{U_2}'\mathbf{V_{22}}\mathbf{U_1}) tr(\mathbf{J}^{-1})$ is large. The motivation of constructing $\mathbf{T_1}(\mathbf{Y})$ comes from the last part of (33). Note that

$$= \left(\sqrt{tr(\sum^*_{\boldsymbol{\Theta}_1})}\ \sqrt{tr(\sum^*_{\boldsymbol{\Theta}_2})}\right)' \begin{pmatrix} tr(\mathbf{U}_1'\mathbf{V}_{11}\mathbf{U}_1)tr(\mathbf{J}^{-1}) & \mathbf{0} \\ \mathbf{0} & tr(\mathbf{U}_2'\mathbf{V}_{22}\mathbf{U}_1)tr(\mathbf{J}^{-1}) \end{pmatrix} \left(\sqrt{tr(\sum^*_{\boldsymbol{\Theta}_1})}\ \sqrt{tr(\sum^*_{\boldsymbol{\Theta}_2})}\right).$$

$$[tr(\sum^*_{\boldsymbol{\Theta}_1})]\,[tr(\mathbf{U}_1'\mathbf{V}_{11}\mathbf{U}_1)]\,[tr(\mathbf{J}^{-1})] + [tr(\sum^*_{\boldsymbol{\Theta}_2})]\,[tr(\mathbf{U}_2'\mathbf{V}_{22}\mathbf{U}_2)]\,[tr(\mathbf{J}^{-1})]$$

Our test statistic $\mathbf{T_1(Y)}$ is the trace of the matrix given in the right side of the above expression. Let us now express $\mathbf{T_1(Y)}$ in terms of the original variables.

$$\begin{aligned} \mathbf{J}^{-1} &= (\mathbf{U}_1'\mathbf{U}_1 + \mathbf{U}_2'\mathbf{U}_2)^{-1} \\ &= (\boldsymbol{Y}_1'(\mathbf{I}-\mathbf{P}_1)\boldsymbol{Y}_1 + \boldsymbol{Y}_2'(\mathbf{I}-\mathbf{P}_2)\boldsymbol{Y}_2),^{-1} \end{aligned} \quad (34)$$

$$tr(\mathbf{U}_1'\mathbf{V}_{11}\mathbf{U}_1) = tr(\boldsymbol{Y}_1'(\mathbf{I}-\mathbf{P}_1)\mathbf{V}_1(\mathbf{I}-\mathbf{P}_1)\boldsymbol{Y}_1), \quad (35)$$

$$tr(\mathbf{U}_2'\mathbf{V}_{22}\mathbf{U}_2) = tr(\boldsymbol{Y}_2'(\mathbf{I}-\mathbf{P}_2)\mathbf{V}_2(\mathbf{I}-\mathbf{P}_2)\boldsymbol{Y}_2), \quad (36)$$

where $\mathbf{P_i}$ is the orthogonal projection matrix onto the column space of $\mathbf{A_i}$, for $i = 1, 2$. Note that $\boldsymbol{Z}_1\boldsymbol{Z}_1' + \mathbf{P}_1 = \mathbf{I}_{N_1}$, and $\boldsymbol{Z}_2\boldsymbol{Z}_2' + \mathbf{P}_2 = \mathbf{I}_{N_2}$. Hence the test statistic for H_0 is

$$\mathbf{T}_1(\boldsymbol{Y}) = tr([(\boldsymbol{Y}_1'(\mathbf{I}-\mathbf{P}_1)\mathbf{V}_1(\mathbf{I}-\mathbf{P}_1)\boldsymbol{Y}_1) + (\boldsymbol{Y}_2'(\mathbf{I}-\mathbf{P}_2)\mathbf{V}_2(\mathbf{I}-\mathbf{P}_2)\boldsymbol{Y}_2)] \times [(\boldsymbol{Y}_1'(\mathbf{I}-\mathbf{P}_1)\boldsymbol{Y}_1 + \boldsymbol{Y}_2'(\mathbf{I}-\mathbf{P}_2)\boldsymbol{Y}_2)^{-1}]). \quad (37)$$

We reject H_{01}, if $\mathbf{T_1(Y)}$ is large. For a given nominal value α, the cut-off point $\mathbf{T}_{1\alpha}$ of the test is determined by the $(1 - \alpha)$100th percentile point of the distribution of $\mathbf{T_1(Y)}$ under the null hypothesis. Note that the test statistic $T_1(Y)$ is unique even though $\mathbf{Z_i}$ is not unique. The results derived above can be expressed in the form of a theorem stated below.

Theorem 1—Consider the mixed-effects model described in (2) with Δ as a fixed-effect matrix and Θ as a random-effect matrix. Let $\mathbf{P_i}$ be the orthogonal projection matrix onto the column space of $\mathbf{A_i}$. To test $H_0 : \Sigma_{\boldsymbol{\Theta}_1} = \Sigma_{\boldsymbol{\Theta}_2} = \mathbf{0}$ against the alternative hypothesis that H_0 is not true, the approximate LBI test rejects H_0 for large values of $\mathbf{T_1(Y)}$.

4. Simulation Study of the LBI Test

To examine the performance of the LBI test described in Theorem 1, we compute its power function via simulation for an unbalanced one-way model with random subject effects. The simulation is carried out in the bivariate case, so Σ_θ is a 2 × 2 matrix. Let K_i denote the number of subjects for the *ith* group ($i = 1, 2$) and n_{ij} denote the number of repeated observations for the *jth* subject nested in the *ith* group. Alternatively, in a clustered design, j might reflect a cluster (e.g., school) and n_{ij} the number of students in the jth school under the ith condition. Write

$$n_i = (N_{i1}, N_{i2}, \cdots, N_{ik_i})' \text{ and } N_i = \sum_{j=1}^{K_i} N_{ij}$$

Now we can verify that $\mathbf{X_i} = \mathbf{diag}(\mathbf{1_{N_{i1}}}, \cdots, \mathbf{1_{N_{ik_i}}})$, $\mathbf{V_i}=\mathbf{X_i}\mathbf{X_i'}$. Construct the matrix $\mathbf{Z_i}$ satisfying the condition that $\boldsymbol{Z_i'}\mathbf{A_i}=\mathbf{0}$, and $\boldsymbol{Z_i'}\boldsymbol{Z_i}=\mathbf{I}$. Let $\mathbf{U_i} = \mathbf{Z_i}\mathbf{Y_i}$ and $\mathbf{V_{ii}}=\boldsymbol{Z_i'}\mathbf{V_i}\boldsymbol{Z_i}$. The test statistic $\mathbf{T_1}(\mathbf{Y})$ in Theorem 1 has the following expression.

$$\mathbf{T_1}(\mathbf{U})=tr\left([(\mathbf{U_1'}\mathbf{V_{11}}\mathbf{U_1})+(\mathbf{U_2'}\mathbf{V_{22}}\mathbf{U_2})]\times[\mathbf{U_1'}\mathbf{U_1}+\mathbf{U_2'}\mathbf{U_2}]^{-1}\right).$$

An Algorithm to determine the cut-off point $\mathbf{T_{1\alpha}}$

- Obtain the fixed-effects covariate matrix $\mathbf{A_i}$, and random-effects covariate matrix $\mathbf{X_i}$.
- Compute $\mathbf{V_i}=\mathbf{X_i}\mathbf{X_i'}$.
- Construct a matrix $\mathbf{Z_i}$ satisfying the condition that $\boldsymbol{Z_i'}\mathbf{A_i}=\mathbf{0}$, and $\boldsymbol{Z_i'}\boldsymbol{Z_i}=\mathbf{I}$.
- Compute $\mathbf{V_{ii}}=\boldsymbol{Z_i'}\mathbf{V_i}\boldsymbol{Z_i}$.
- Generate $\mathbf{U_i} = \mathbf{Z_i}\mathbf{Y_i}$ for 10,000 times following the distribution (under the null hypothesis) that $Vec(\mathbf{U_i'})$ follows a multivariate normal with mean vector $\mathbf{0}$, and covariance matrix $\mathbf{I_{(N_i-r_i)}} \otimes \mathbf{I_p}$.
- For each generated data, compute $\mathbf{T_1}(\mathbf{U})$ following the above expression of $\mathbf{T_1}(\mathbf{U})$.
- Obtain the $(1-\alpha)100th$ percentile of the computed $\mathbf{T_1}(\mathbf{U})$ values, and call it $\mathbf{T_{1\alpha}}$.

To compute the simulated power of the test we generate $Vec(\mathbf{U}')$ for 10,000 times from a multivariate normal distribution with mean vector $\mathbf{0}$, and covariance matrix given in equation (7) on page 6 (considering $\Sigma = \mathbf{I}$). For each generated data, we compute $\mathbf{T_1}(\mathbf{U})$, and compare it with $\mathbf{T_{1\alpha}}$. The simulated power is the number of times $\mathbf{T_1}(\mathbf{U})$ exceeds the cut-off point $\mathbf{T_{1\alpha}}$ divided by 10,000.

The following models are considered for the simulation study based on test 3 on page8:

1. $K_1 = 4, K_2 = 5$

   $n_1 = (4, 4, 4, 5)'$, $n_2 = (4, 4, 4, 4, 3)'$.

2. $K_1 = 5, K_2 = 6$

   $n_1 = (4, 4, 4, 5, 7)'$, $n_2 = (4, 4, 4, 6, 7, 7)'$.

We used $\Sigma = \mathbf{I}$ and $\alpha = .05$. The covariance matrices for the random-effects, i.e., Σ_{θ_i}'s were varied for the following combinations,

$$\sum\nolimits_{\Theta_i}=a_i\mathbf{I},\ \sum\nolimits_{\Theta_i}=a_i\begin{pmatrix}\mathbf{2} & \mathbf{1}\\ \mathbf{1} & \mathbf{1}\end{pmatrix},\ \text{and}\ \sum\nolimits_{\Theta_i}=a_i\begin{pmatrix}\mathbf{1} & \mathbf{1}\\ \mathbf{1} & \mathbf{1}\end{pmatrix},$$

for $a_i = 2, 1, .5, .05, 0$. The cut-off points were determined based on 10,000 simulations under the assumption that $\Sigma_{\theta_1} = \Sigma_{\theta_2} = \mathbf{0}$.

Remark 1—We notice that the power responds sharply to changes in the sample sizes (compare Tables 1i with Tables 2i, for i=I, II, III, IV) and also to the alternative parameterization of the random-effects covariance matrices. The right most bottom number of all tables indicates that the nominal type I error rate is achieved in all cases.

5. Comparison of LBI and the LRT Chi – square Test

In an attempt to better understand the statistical properties of our proposed LBI test, we compared the power of our test with the traditional LR χ^2 test. To do this, we simulated a dataset consisting of $k = 6$ and 10 subjects per group $n = 4$ repeated measurements, for $p = 2$ response variables (*i.e*., bivariate), and one random-effect (*i.e*., random intercept) per response variable. In this simple illustration, we selected the error matrix $\Sigma = \mathbf{I}$, and the random-effects covariance matrices $\Sigma_{\theta_1} = \Sigma_{\theta_2} = a_1\mathbf{I}$, and $\sum_{\Theta_1}=\sum_{\Theta_2}=a_1\begin{pmatrix}\mathbf{1} & \mathbf{1}\\ \mathbf{1} & \mathbf{1}\end{pmatrix}$ for $a_1 = 0.0, \ldots, 1.0$. In this way, we can determine the power of a test that the random-effect variance is equal to zero, and compare the power for the LBI and LR χ^2 tests. Note that when $a_1 = 0$, the data are consistent with the null hypothesis, and the power reflects the Type I error rate of the two tests. The Type I error rate was set a priori to 0.05. The simulation study was based on 10,000 replications.

The tables reveal that the LBI test is considerably more powerful than the LR χ^2 test, due to its' small sample properties. In Table 3II, the LBI test produces power of 0.8 for a value of *a* close to .5, whereas the LR χ^2 test produces the same power of 0.8 for a value of *a* close to 2. These findings reveal that the development of new small sample alternatives to traditional LR χ^2 tests for variance components will lead to tests that control Type I error rates at the nominal level, and have increased statistical power.

6. The Example

Sun et. al.,(2003) observed that after log transformation, height and weight have approximately normal distributions. Furthermore, the log transformed heights and weights are approximately linear with respect to age between one year and ten years. We now use these transformed data to illustrate our method. See Table 4 for complete data.

Sun et. al., (2003) proposed the following random-intercept model for the *k*th time point of the *j*th subject belonging to the *i*th group.

$$\mathbf{y}_{ijk}=\mu_{\mathbf{i}}+t_{ijk}\beta_{\mathbf{i}}+\theta_{ij}+\mathbf{e}_{ijk},$$

where $\mathbf{y}_{ijk}$, $\mu_{\mathbf{i}} = (\mu_{i1}\ \mu_{i2})$, $\beta_{\mathbf{i}} = (\beta_{i1}\ \beta_{i2})$, $\theta_{ij} = (\theta_{ij1}\ \theta_{ij2})$, and $\mathbf{e}_{ijk}$ are all 1 × 2 vectors and the scalar number t_{ijk} represents the correspnding time value. The model in the matrix notation for the *j*th subject belonging to the *i*th group is

$$\boldsymbol{Y}_{ij}=\mathbf{1}\mu_{\mathbf{i}}+\mathbf{T}_{ij}\beta_{\mathbf{i}}+\mathbf{U}_{ij}+\mathbf{E}_{ij},$$

where $\mathbf{U}_{ij} = (\theta_{ij}, \cdots, \theta_{ij})'$. We assume that the rows of $\mathbf{E}_{ij}s$ are independent and each row follows a bivariate normal distribution with mean 0 and covariance matrix Σ, also θ_{ij} and the rows of $\mathbf{E}_{ij}s$ are independently distributed. We assume that $\theta_{ij} \sim N(0, \Sigma_{\theta_i})$. Note that $p = 2$, as the number of columns corresponding to height and weight of the observation matrix $\mathbf{Y_i}$ is 2. We assume that both the intercept $\mu_{\mathbf{i}}$ and the slope $\beta_{\mathbf{i}}$ of the model are fixed for both the components height and weight. The covariate matrix $\mathbf{A_i}$ in (2) consists of two columns; the elements of the first column are all 1, and those of the second column are t_{ijk}, i.e., $\mathbf{A_i} = (\mathbf{1}_{\mathbf{N_i}}$

$\mathbf{T_i}$). Hence the number of columns of the covariate matrix $\mathbf{A_i}$ is 2, i.e., $r_1 = r_2 = 2$. The fixed parameter matrix $\mathbf{\Delta_i}$ in (2) has the following expression.

$$\mathbf{\Delta_i}=\begin{pmatrix} \mu_\mathbf{i} \\ \beta_\mathbf{i} \end{pmatrix} \text{ is a } 2 \times 2 \text{ matrix.}$$

The random effect θ_{ij} of subject *j* nested within the *ith* group represents the deviation from the mean intercept $\mu_\mathbf{i}$. The covariance matrix $\mathbf{X_i}$ for the random effects has the following expression.

$$\mathbf{X_i}=\begin{pmatrix} \mathbf{1}_{N_{i1}} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_{N_{i2}} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1}_{N_{i3}} & \cdots & \mathbf{0} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{1}_{N_{i15}} \end{pmatrix}, \quad (38)$$

where $\mathbf{1}_{N_{il}}$ is a column vector of order N_{il} with all elements 1. Hence s_i, the number of columns of the design matrix of the random effects $\mathbf{X_i}$ is 15, i.e., $s_1 = s_2 = 15$. There were 78 observations from 15 boys and 82 observations from 15 girls. Hence $N_1 = 78$, $N_2 = 82$, and $n_1 = n_2 = 15$.

Note that

$$\mathbf{V_i}=\mathbf{X_i}\mathbf{X_i}'=\begin{pmatrix} \mathbf{J}_{N_{i1}} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_{N_{i2}} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{J}_{N_{i3}} & \cdots & \mathbf{0} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{J}_{N_{i15}} \end{pmatrix}. \quad (39)$$

We const ruct the matrix $\mathbf{Z_i}$ satisfying the condition that $Z_\mathbf{i}'\mathbf{A_i}=\mathbf{0}$, and $Z_\mathbf{i}'Z_\mathbf{i}=\mathbf{I}$. Let $\mathbf{U_i} = \mathbf{Z_i}\mathbf{Y_i}$ and $\mathbf{V_{ii}}=Z_\mathbf{i}'\mathbf{V_i}Z_\mathbf{i}$. Following Theorem 1, we compute $\mathbf{T_1}(\mathbf{Y}) = \mathbf{6.8054742}$, with the cut-off point $\mathbf{T_{1\alpha}} = \mathbf{2.5561}$, for $\alpha = .05$. Hence we reject the null hypothesis $H_0 : \Sigma_{\theta_1} = \Sigma_{\theta_2} = \mathbf{0}$. Note that the cut-off point is based on 10,000 simulations.

7. Discussion

Statistical hypothesis testing for univariate and multivariate mixed-effects regression models have lagged behind general statistical research and application in this area. In this paper, we have derived an optimal test for testing the significance of variance components in a multivariate mixed-effects regression model. Although we have motivated the problem by using a two group comparison of growth curves (*i.e*., comparison of boys and girls in terms of log-linear rate of growth between the ages of one and ten years), the method can be easily generalized for any number of groups, including the simple case of testing the significance of variance components in a single sample. The focus of our test is on determining if the variance component(s) is non-zero (*i.e*., does the random effect(s) need to be included in the model). This problem is not adequately addressed by traditional tests, such as the likelihood ratio chi-square statistic, because under the null hypothesis, the parameter lies on the boundary of the parameter space, therefore violating the conditions that are required for the test statistic to have an asymptotic chi-square distribution. Our optimal test does not suffer from this problem, and it enjoys optimal small sample properties. The net result is a test which achieves its intended nominal Type I error rate, but provides markedly greater statistical power relative to the traditionally used, yet misapplied, likelihood-ratio test. The practical disadvantage of our method is that it requires simulation in order to obtain the critical points of the distribution of the test statistic. To this end we have provided a relatively straightforward algorithm for computation of the statistic and its' corresponding critical value.

Note that testing the significance of variance components is only one of several hypothesis testing problems for which further statistical research is required. For example, the variance components may be nonzero, but unequal. Our test does not address this problem, and it is unclear as to whether or not an optimal test for this problem exists. Alternative small sample approximate tests should be investigated. In addition, small sample optimal and/or approximate tests for fixed-effects in the general mixed-effects regression model are also needed. This is true for both univariate and multivariate problems. Finally, we have only considered a linear mixed-effects regression model. Optimal tests for both random and fixed-effects in nonlinear mixed-effects regression models (*e.g*., mixed-effects logistic, probit, Poisson … regression models) are also needed.

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Table 1.I

Power for the LBI tests based on 10,000 simulations for testing $H_0 : \Sigma_{\theta_1} = \Sigma_{\theta_2} = \mathbf{0}$ in the one-way models for $\alpha = .05$, and $\Sigma = \mathbf{I}$. $n_1 = (4, 4, 4, 5)'$, $n_2 = (4, 4, 4, 4, 3)'$, $\Sigma_{\theta_1} = a_1\mathbf{I}$, $\Sigma_{\theta_2} = a_2\mathbf{I}$.

	a_1				
a_2	**2**	**1**	**0.5**	**0.05**	**0**
2	.9999	.9968	.9902	.9818	.9890
1	.9959	.9832	.9533	.9048	.9382
0.5	.9864	.9445	.8577	.6888	.7878
0.05	.9557	.8386	.6066	.1322	.1333
0	.9543	.8297	.5863	.0857	.0501

Table 1.II

Power for the LBI tests based on 10,000 simulations for testing $H_0 : \Sigma_{\theta_1} = \Sigma_{\theta_2} = \mathbf{0}$ in the one-way models for $\alpha = .05$, $\Sigma = \mathbf{I}$. $n_1 = (4, 4, 4, 5)'$, $n_2 = (4, 4, 4, 4, 3)'$, $\Sigma_{\theta_1} = a_1\mathbf{I}$, $\sum_{\Theta_2} = a_2 \begin{pmatrix} \mathbf{2} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} \end{pmatrix}$.

	***a*₁**				
***a*₂**	**2**	**1**	**0.5**	**0.05**	**0**
2	.9997	.9981	.9945	.9896	.9890
1	.9974	.9888	.9718	.9406	.9382
0.5	.9906	.9622	.9045	.7949	.7878
0.05	.9587	.8462	.6213	.1652	.1333
0	.9543	.8297	.5863	.0857	.0501

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Table 1.III

Power for the LBI tests based on 10,000 simulations for testing $H_0 : \Sigma_{\theta_1} = \Sigma_{\theta_2} = \mathbf{0}$ in the one-way models for $\alpha = .05$, $\Sigma = \mathbf{I}$. $n_1 = (4, 4, 4, 5)'$, $n_2 = (4, 4, 4, 4, 3)'$, $\Sigma_{\theta_1} = a_1\mathbf{I}$, $\sum_{\Theta_2} = a_2 \begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} \end{pmatrix}$.

	a_1				
a_2	**2**	**1**	**0.5**	**0.05**	**0**
2	.9974	.9897	.9751	.9437	.9422
1	.9919	.9677	.9242	.8360	..8302
0.5	.9836	.9331	.8347	.6406	.6274
0.05	.9566	.8400	.6081	.1355	.1027
0	.9543	.8297	.5863	.0857	.0501

Table 1.IV

Power for the LBI tests based on 10,000 simulations for testing $H_0 : \Sigma_{\theta_1} = \Sigma_{\theta_2} = \mathbf{0}$ in the one-way models for $\alpha = .05$, $\Sigma = \mathbf{I}$. $n_1 = (4, 4, 4, 5)'$, $n_2 = (4, 4, 4, 4, 3)'$, $\sum_{\Theta_1} = a_1 \begin{pmatrix} \mathbf{2} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} \end{pmatrix}$, $\sum_{\Theta_2} = a_2 \begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} \end{pmatrix}$.

	a_1				
a_2	**2**	**1**	**0.5**	**0.05**	**0**
2	.9976	.9920	.9811	.9455	.9422
1	.9937	.9766	.9432	.8408	.8302
0.5	.9870	.9535	.8825	.6503	.6274
0.05	.9685	.8855	.7138	.1573	.1027
0	.9665	.8783	.6967	.1091	.0501

Table 2.I

Power for the LBI tests based on 10,000 simulations for testing $H_0 : \Sigma_{\theta_1} = \Sigma_{\theta_2} = \mathbf{0}$ in the one-way models for $\alpha = .05$, $\Sigma = \mathbf{I}$. $n_1 = (4, 4, 4, 5, 7)'$, $n_2 = (4, 4, 4, 6, 7, 7)'$, $\Sigma_{\theta_1} = a_1\mathbf{I}$, $\Sigma_{\theta_2} = a_2\mathbf{I}$.

	a_1				
a_2	**2**	**1**	**0.5**	**0.05**	**0**
2	.9999	.9999	.9997	.9991	.9991
1	.9996	.9976	.9934	.9806	.9802
0.5	.9977	.9843	.9492	.8736	.8712
0.05	.9828	.8985	.6673	.1636	.1487
0	.9805	.8875	.6318	.0729	.0501

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Table 2.II

Power for the LBI tests based on 10,000 simulations for testing $H_0 : \Sigma_{\theta_1} = \Sigma_{\theta_2} = \mathbf{0}$ in the one-way models for $\alpha = .05$, $\Sigma = \mathbf{I}$. $n_1 = (4, 4, 4, 5, 7)'$, $n_2 = (4, 4, 4, 6, 7, 7)'$, $\Sigma_{\theta_1} = a_1 \mathrm{I}$, $\sum_{\Theta_2} = a_2 \begin{pmatrix} \mathbf{2} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} \end{pmatrix}$.

	***a*₁**				
a_2	**2**	**1**	**0.5**	**0.05**	**0**
2	.9999	.9998	.9995	.9992	.9992
1	.9999	.9990	.9959	.9892	.9890
0.5	.9984	.9919	.9700	.9267	.9251
0.05	.9844	.9057	.6939	.2311	.2122
0	.9805	.8875	.6318	.0739	.0500

Table 2.III

Power for the LBI tests based on 10,000 simulations for testing $H_0 : \Sigma_{\theta_1} = \Sigma_{\theta_2} = \mathbf{0}$ in the one-way models for $\alpha = .05$, $\Sigma = \mathbf{I}$. $n_1 = (4, 4, 4, 5, 7)'$, $n_2 = (4, 4, 4, 6, 7, 7)'$, $\Sigma_{\theta_1} = a_1\mathbf{I}$, $\sum_{\Theta_2} = a_2 \begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} \end{pmatrix}$.

	a_1				
a_2	**2**	**1**	**0.5**	**0.05**	**0**
2	.9997	.9986	.9937	.9847	.9841
1	.9984	.9924	.9765	.9394	.9377
0.5	.9962	.9777	.9249	.8110	.8062
0.05	.9824	.8991	.6717	.1742	.1537
0	.9805	.8875	.6318	.0739	.0501

Table 2.IV

Power for the LBI tests based on 10,000 simulations for testing $H_0 : \Sigma_{\theta_1} = \Sigma_{\theta_2} = \mathbf{0}$ in the one-way models for $\alpha = .05$, $\Sigma = \mathbf{I}$. $n_1 = (4, 4, 4, 5, 7)'$, $n_2 = (4, 4, 4, 6, 7, 7)'$, $\sum_{\Theta_1} = a_1 \begin{pmatrix} \mathbf{2} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} \end{pmatrix}$, $\sum_{\Theta_2} = a_2 \begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} \end{pmatrix}$.

	***a*₁**				
a_2	**2**	**1**	**0.5**	**0.05**	**0**
2	.9997	.9985	.9937	.9847	.9841
1	.9994	.9953	.9765	.9394	.9377
0.5	.9978	.9875	.9249	.8110	.8062
0.05	.9982	.9362	.6717	.1742	.1537
0	.9866	.9281	.6318	.0739	.0501

Table 3.I

Power for the LBI and LR tests based on 10,000 simulations for testing $H_0 : \Sigma_{\theta_1} = \Sigma_{\theta_2} = \mathbf{0}$ in one-way models for $a = .05$, $\Sigma = \mathbf{I}$. $k = 6$, $n_1 = n_2 = (4, 4, 4, 4, 4, 4)'$, $H_1: \sum_{\Theta_1} = \sum_{\Theta_2} = a \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}$.

a	0	.01	.03	.05	.1	.2	.5	1	2
Power of LBI	.0535	.0665	.0987	.1384	.2562	.5194	.9158	.9942	.9999
Power of LRT	.0500	.0551	.0625	.0636	.0849	.1260	.3017	.5935	.8984

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Table 3.II

Power for the LBI and LR tests based on 10,000 simulations for testing $H_0 : \Sigma_{\theta_1} = \Sigma_{\theta_2} = \mathbf{0}$ in one-way models for $a = .05$, $\Sigma = \mathbf{I}$. $k = 6$, $n_1 = n_2 = (4, 4, 4, 4, 4, 4)'$, $H_1{:}\sum_{\Theta_1}=\sum_{\Theta_2}=a\begin{pmatrix}\mathbf{1} & \mathbf{1}\\ \mathbf{1} & \mathbf{1}\end{pmatrix}$.

a	0	.01	.03	.05	.1	.2	.5	1	2
Power of LBI	.0501	.0611	.0909	.1252	.2265	.4287	.8041	.9595	.9957
Power of LRT	.0500	.0553	.0576	.0682	.0850	.1323	.3245	.6150	.8766

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Table 3.III

Power for the LBI and LR tests based on 10,000 simulations for testing $H_0 : \Sigma_{\theta_1} = \Sigma_{\theta_2} = \mathbf{0}$ in one-way models for $\alpha = .05$, $\Sigma = \mathbf{I}$. $k = 10$, $n_1 = n_2 = (4, 4, 4, 4, 4, 4, 4, 4, 4, 4)'$, $\sum_{\Theta_1} = \sum_{\Theta_2} = a \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}$.

a	0	.01	.03	.05	.1	.2	.5	1	2
Power of LBI	.0500	.0671	.1147	.1737	.3588	.7199	.9891	.9999	.9999
Power of LRT	.0497	.0527	.0592	.0683	.0875	.1553	.4170	.7799	.9817

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Table 3.IV

Power for the LBI and LR tests based on 10,000 simulations for testing $H_0 : \Sigma_{\theta_1} = \Sigma_{\theta_2} = \mathbf{0}$ in one-way models for $\alpha = .05$, $\Sigma = \mathbf{I}$. $k = 10$, $n_1 = n_2 = (4, 4, 4, 4, 4, 4, 4, 4, 4, 4)'$, $H_1 : \sum_{\Theta_1} = \sum_{\Theta_2} = a \begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} \end{pmatrix}$.

a	0	.01	.03	.05	.1	.2	.5	1	2
Power of LBI	.0501	.0654	.1112	.1653	.3220	.6300	.9574	.9977	.9999
Power of LRT	.0500	.0546	.0632	.0671	.0954	.1623	.4394	.8019	.9765

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Table 4.I

Sample Data for Boys

Obs	Id	Y1	T	Y2
1	1	2.36085	1.02192	4.34381
2	1	2.42480	1.47123	4.40672
3	1	2.63906	2.56438	4.53260
4	1	2.76001	2.94795	4.53260
5	1	2.85071	4.04110	4.66344
6	1	2.99573	5.61370	4.76217
7	1	2.31419	8.36986	4.89035
8	2	2.77882	2.03014	4.54329
9	2	2.90690	2.70137	4.60517
10	2	3.07269	4.13151	4.70953
11	2	3.36730	6.83562	4.85981
12	2	3.43399	7.66301	4.89784
13	3	2.43493	1.71507	4.42485
14	3	2.60269	1.76986	4.52721
15	3	2.73437	4.07123	4.62497
16	3	2.94969	5.89589	4.73180
17	3	3.08191	7.74247	4.80402
18	4	2.43361	1.16438	4.41280
19	4	2.51447	1.35616	4.43082
20	4	2.61007	1.73973	4.49981
21	4	2.83908	2.69863	4.61611
22	4	2.88480	3.15890	4.64535
23	4	2.97041	4.00822	4.70502
24	4	3.12236	5.78630	4.79165
25	4	3.29584	7.69315	4.88280
26	4	3.37759	8.79452	4.93447
27	5	2.56495	1.60274	4.45435
28	5	2.93386	4.08219	4.86213
29	5	3.08191	5.64932	4.78749
30	5	3.26576	7.66301	4.89784
31	6	2.52413	1.67671	4.41884
32	6	2.83908	4.10137	4.62497
33	6	2.99573	5.70959	4.71850
34	6	3.32504	7.66027	4.83628
35	7	2.28034	1.18082	4.32413
36	7	2.35423	1.23014	4.34381
37	7	2.83908	4.30137	4.64439
38	7	3.01553	5.64384	4.74057
39	7	3.19867	7.62192	4.82028

Obs	Id	Y1	T	Y2
40	7	3.40120	9.63836	4.89035
41	8	2.32239	1.06027	4.29046
42	8	2.41948	1.82740	4.39445
43	8	2.57261	2.80548	4.51086
44	8	2.70136	4.05205	4.57471
45	8	2.89037	5.83562	4.71850
46	8	3.11795	7.67671	4.79579
47	9	2.62467	1.88493	4.49981
48	9	2.97041	4.18082	4.68213
49	9	3.21084	6.10411	4.85981
50	9	3.36730	7.64932	4.89784
51	10	2.86790	4.06575	4.69135
52	10	3.02042	5.72603	4.79579
53	10	3.12676	6.89589	4.85203
54	10	3.25810	7.64658	4.88280
55	11	2.60195	1.61918	4.48864
56	11	2.83908	2.99726	4.60517
57	11	2.88480	3.99726	4.67749
58	11	2.95491	5.62740	4.76217
59	11	2.99573	7.71507	4.86753
60	12	2.47654	1.19726	4.38826
61	12	2.58022	1.83288	4.45435
62	12	2.86220	4.32055	4.67283
63	12	3.25810	7.62740	4.85203
64	13	2.50960	1.51781	4.45435
65	13	2.83908	4.09863	4.66344
66	13	3.10009	6.13425	4.78332
67	13	3.19458	7.74795	4.85203
68	14	2.57261	1.69041	4.44852
69	14	2.74084	2.34247	4.52721
70	14	2.91777	4.01096	4.64439
71	14	3.15700	5.66027	4.75359
72	14	3.28091	6.83014	4.80402
73	14	3.39451	7.58904	4.84419
74	15	2.40695	1.80000	4.44852
75	15	2.56495	2.77534	4.54860
76	15	2.71469	4.02192	4.62006
77	15	2.94444	5.82466	4.73620
78	15	3.10459	7.61370	4.82028

Obs=Observation, Id=Identification, Y1=log(Weight(kg)), T=Age, Y2=log(Height(cm))

Table 4.II

Sample Data for Girls

Obs	Id	Y1	T	Y2
1	1	2.53370	1.54795	4.39445
2	1	2.61740	2.00822	4.44852
3	1	2.74084	2.98630	4.56435
4	1	2.86220	4.03014	4.64439
5	1	3.04452	6.01644	4.77068
6	1	3.21888	7.43562	4.83231
7	2	2.22462	1.00548	4.31749
8	2	2.58022	2.78904	4.55388
9	2	2.61007	4.02740	4.65396
10	2	2.90142	6.07397	4.76217
11	2	3.00072	7.38904	4.83628
12	3	2.48491	1.47397	4.39445
13	3	2.70805	2.58904	4.54329
14	3	2.87356	4.07123	4.64439
15	3	2.99573	5.63562	4.73180
16	3	3.10459	7.42192	4.80402
17	4	2.56955	1.75616	4.44265
18	4	2.74727	2.96438	4.54329
19	4	2.92852	4.07397	4.61512
20	4	3.09104	6.10959	4.74057
21	4	3.18221	7.41918	4.80402
22	5	2.36085	1.65753	4.39445
23	5	2.75366	4.12603	4.63957
24	5	2.97041	5.62466	4.74493
25	5	3.09104	7.41096	4.85203
26	6	2.44235	1.58356	4.40060
27	6	2.63906	2.54247	4.47734
28	6	2.73437	4.05479	4.61512
29	6	2.89037	5.74521	4.70953
30	6	3.32143	7.43836	4.78749
31	7	2.72130	2.50411	4.52179
32	7	2.80336	2.83014	4.54860
33	7	2.97041	4.01918	4.65396
34	7	3.07731	5.87945	4.75359
35	7	3.25810	7.29041	4.82831
36	8	2.67415	1.68493	4.49981
37	8	2.83908	2.95068	4.64439
38	8	2.99573	3.95890	4.73620
39	8	3.18635	5.36438	4.84419

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Obs	Id	Y1	T	Y2
40	8	3.25810	5.86757	4.87520
41	8	3.33220	6.53973	4.92725
42	8	3.43399	7.29863	4.93447
43	8	3.52636	8.41096	4.97673
44	9	2.45101	1.41096	4.38203
45	9	2.541606	1.90959	4.41280
46	9	2.70136	2.88767	4.51634
47	9	2.83908	3.98356	4.60517
48	9	3.03013	5.66849	4.74493
49	9	3.26576	7.33425	4.82028
50	10	2.39790	1.64384	4.44265
51	10	2.75366	3.18356	4.59512
52	10	2.81541	3.93425	4.65396
53	10	3.07731	5.71781	4.76217
54	10	3.28840	7.28219	4.86368
55	11	2.35138	1.03288	4.31749
56	11	2.58022	1.73699	4.44265
57	11	2.76632	2.48493	4.53796
58	11	2.97553	3.97808	4.68213
59	11	3.21487	6.03288	4.81218
60	11	3.33392	7.33425	4.88280
61	12	2.37955	1.04384	4.38203
62	12	3.06805	5.30411	4.74493
63	12	3.15700	6.22466	4.80402
64	12	3.22684	7.25205	4.85981
65	13	2.35802	1.07397	4.34381
66	13	2.45101	1.52055	4.41884
67	13	2.82138	3.12603	4.58497
68	13	2.93916	4.05479	4.67283
69	13	3.04452	5.51233	4.74057
70	13	3.33220	7.29863	4.83231
71	14	2.36368	1.34247	4.38203
72	14	2.48491	1.92877	4.44265
73	14	2.60269	2.37808	4.48864
74	14	2.52573	2.66575	4.53260
75	14	2.77259	3.96986	4.62497
76	14	2.85071	4.62192	4.68213
77	14	3.04452	5.94521	4.76217
78	14	3.23080	7.24932	4.82028
79	15	2.62467	1.66575	4.46591
80	15	2.91777	2.88219	4.67283
81	15	2.91977	4.24479	4.67283

Obs	Id	Y1	T	Y2
82	15	3.23868	7.22466	4.85981

Obs=Observation, Id=Identification, Y1=log(Weight(kg)), T=Age, Y2=log(Height(cm))